Dodge the Lasers! Analysis

Colin Vandenhof

May 2021

A Beatty sequence is a sequence of the form: $\lfloor\theta\rfloor, \lfloor 2\theta\rfloor, \lfloor 3\theta\rfloor, ...$, where θ is a positive irrational number.

The Rayleigh theorem states that if α and β are positive irrational numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then both of their respective Beatty sequences contain all of the positive integers without repetition[1].

Let $F(\alpha, n)$ give the sum of the first n terms of the Beatty sequence with value α:

$$F(\alpha, n) = \sum_{k=1}^{n} \lfloor k\theta \rfloor$$

Let x be the nth term, $x = \lfloor n\alpha \rfloor$. Then by the Rayleigh theorem,

$$F(\alpha, n) + F(\beta, m) = \sum_{k=1}^{x} k = \frac{x(x+1)}{2}$$

where m is the greatest integer such that $\lfloor m\beta \rfloor < x$, so $m = \left\lfloor \frac{x}{\beta} \right\rfloor$.

m can be re-expressed in terms of n and α as follows:

$$m = \left\lfloor \frac{x}{\beta} \right\rfloor = \left\lfloor \frac{(\alpha-1)x}{\alpha} \right\rfloor = x + \left\lfloor -\frac{x}{\alpha} \right\rfloor = x - \left\lceil \frac{x}{\alpha} \right\rceil$$

$$= \lfloor n\alpha \rfloor - \left\lceil \frac{\lfloor n\alpha \rfloor}{\alpha} \right\rceil = \lfloor n\alpha \rfloor - n = \lfloor n(\alpha - 1) \rfloor$$

[1] https://mathworld.wolfram.com/BeattySequence.html

Hence, we derive a recurrence for $F(\alpha, n)$:

$$\begin{aligned} F(\alpha, n) &= \frac{x(x+1)}{2} - F(\beta, m) \\ &= \frac{\lfloor n\alpha \rfloor (\lfloor n\alpha \rfloor + 1)}{2} - F(\beta, m) \\ &= \frac{(\lfloor n(\alpha-1) \rfloor + n)(\lfloor n(\alpha-1) \rfloor + n + 1)}{2} - F(\beta, m) \\ &= \frac{(m+n)(m+n+1)}{2} - F(\beta, m) \end{aligned}$$

With $\alpha = \sqrt{2}$ and $\beta = \sqrt{2} + 2$, we get:

$$\begin{aligned} F(\sqrt{2}, n) &= \frac{(m+n)(m+n+1)}{2} - F(\sqrt{2}+2, m) \\ &= \frac{(m+n)(m+n+1)}{2} - F(\sqrt{2}, m) - 2\sum_{k=1}^{m} k \\ &= \frac{(m+n)(m+n+1)}{2} - F(\sqrt{2}, m) - m(m+1) \\ &= mn + \frac{n(n+1)}{2} - \frac{m(m+1)}{2} - F(\sqrt{2}, m) \end{aligned}$$

This suggests a simple recursive algorithm for computing $F(\alpha, n)$. To accurately compute $m = \left\lfloor n(\sqrt{2}-1) \right\rfloor$ for large values of n (i.e. up to $n = 10^{100}$), we must avoid floating point operations. Instead, we can precompute $c = \lfloor (\sqrt{2}-1) * 10^{100} \rfloor = \lfloor \sqrt{2 * 10^{200}} \rfloor - 10^{100}$ using an integer square root algorithm[2] and then compute m as $\frac{c*n}{10^{100}}$.

[2] `https://en.wikipedia.org/wiki/Integer_square_root`