Disorderly Escape Analysis

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We consider the group $G = S_w \times S_h$, where S_w and S_h are symmetric groups. Let $Y = \{1, 2, ..., s\}$ (the set of states) and $X = W \times H = \{1, 2, ..., w\} \times \{1, 2, ..., h\}$ (the set of grid indices). Then Y^X is the set of functions $X \to Y$ (mappings from grid index to state).

By Pólya enumeration theorem, we have that:

$$|Y^X/G| = \frac{1}{|G|} \sum_{g \in G} s^{c(g)}$$

where $|Y^X/G|$ is the number of orbits under G and $c(g)$ is the number of cycles of group element g.

This result can be rewritten as a cycle index polynomial:

$$|Y^X/G| = Z(G, s, s, ..., s)$$

The cycle index polynomial of symmetric group S_n [1] is given by:

$$Z(S_n, x_1, ..., x_n) = \frac{1}{n!} \sum_{j_1 + 2j_2 + ... + nj_n = n} \frac{n! \prod_{k=1}^{n} x_k^{j_k}}{\prod_{k=1}^{n} k^{j_k} j_k!}$$

The cycle index polynomial of the product $S_w \times S_h$ [2] is given by:

$$\begin{aligned} Z(S_w \times S_h, x_1, ..., x_{wh}) &= Z(S_w, x_1, ..., x_w) ※ Z(S_h, x_1, ..., x_h) \\ &= \sum (a_{i_1 i_2 ... i_w} b_{j_1 j_2 ... j_h} \prod_{\substack{1 \leq l \leq w \\ 1 \leq m \leq h}} x_{[l,m]}^{i_l j_m (l,m)}) \end{aligned}$$

[1] https://mathworld.wolfram.com/CycleIndex.html
[2] https://www.sciencedirect.com/science/article/pii/0012365X9390015L

where (l,m) and $[l,m]$ denote the greatest common divisor and least common multiple respectively.

Putting these results together, the final formula for $|Y^X/G|$ can be derived:

$$
\begin{aligned}
|Y^X/G| &= Z(S_w \times S_h, s, s, ..., s) \\
&= \sum (a_{i_1 i_2 \ldots i_w} b_{j_1 j_2 \ldots j_h} \prod_{\substack{1 \le l \le w \\ 1 \le m \le h}} s^{i_l j_m (l,m)}) \\
&= \frac{1}{w!h!} \sum_{\substack{i_1+2i_2+\ldots+wi_w=w \\ j_1+2j_2+\ldots+hj_h=h}} \frac{w!}{\prod_{k=1}^{w} k^{i_k} i_k!} \frac{h!}{\prod_{k=1}^{h} k^{j_k} j_k!} s^{\sum_{\substack{1 \le l \le w \\ 1 \le m \le h}} i_l j_m (l,m)}
\end{aligned}
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